

The two-dimensional three-body problem in a strong magnetic field is integrable

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The problem of N particles interacting through pairwise central forces is notoriously intractable for $N \geq 3$. Some quite remarkable specific cases have been solved in one dimension, whereas higher-dimensional exactly solved systems involve velocity-dependent or many-body forces. Here we show that the guiding center approximation—valid for charges moving in two dimensions in a strong constant magnetic field—simplifies the three-body problem for an arbitrary interparticle interaction invariant under rotations and translations and makes it solvable by quadratures. This includes a broad variety of special cases, such as that of three particles interacting through arbitrary pairwise central potentials. A spinorial representation for the system is introduced, which allows a visualization of its phase space as the corresponding Bloch sphere as well as the identification of a Berry-Hannay rotational anholonomy. Finally, a brief discussion of the quantization of the problem is presented.

Keywords: three-body problem, guiding center dynamics, magnetic field

It is only in a few select cases that the N -body problem, with $N \geq 3$, is known to be integrable. In arbitrary dimensions, the best known example is that of N particles interacting through linear forces, first solved by Newton [1]. In one dimension, there are several cases, such as that of N particles interacting through an r^{-2} potential. This was solved by Calogero [2] and Marchioro [3] and for $N = 3$ (but see also [4] for earlier related results) and by Calogero [5] and Sutherland [6] for the case of the quantum system with arbitrary N and all interaction strengths equal; the corresponding classical problem was solved by Moser [7]. While integrable N -body problems can also be found in two and three dimensions, these remarkable results generally involve somewhat peculiar features, such as velocity-dependent forces, many-body interactions, or Hamiltonians that are not of the usual form of the sum of kinetic and potential energy. The reader will find an extensive treatment and many references in [8] and more recent results in [9].

The aim of this letter is to present a general class of integrable systems of a rather different nature. On the one hand, they admit a broad class of interactions be-

tween the three particles: any force defined by a rotationally and translationally invariant potential is allowed. This includes in particular the case in which the particles interact via arbitrary pairwise central potentials. On the other hand, they are explicitly limited to the case of three particles moving in two dimensions. The feature that makes the problem solvable is that the particles are charged with the same charge e in the presence of a strong constant magnetic field B . The latter induces a rapid circular motion of particle i of radius $r_i = m_i v_i / |eB|$, where m_i and v_i are the mass and speed of the particle respectively (we use $c = 1$ throughout). If the field is sufficiently strong, the r_i become negligible relative to any other length-scales of the problem. With this fundamental assumption, the effective Hamiltonian system describing the secular motion becomes integrable by virtue of the symmetries of the interaction potential.

Let us turn to a detailed description of the system. Let \vec{q}_i and \vec{p}_i be the position and canonical momentum vectors of particle $i = 1, 2, 3$, with components $q_{i,\alpha}$ and $p_{i,\alpha}$ respectively ($\alpha = 1, 2$), and suppose the exact Hamiltonian of the system is

$$H(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^3 \frac{1}{2m_i} \left[(p_{i,1} - eB q_{i,2})^2 + p_{i,2}^2 \right] + V(\vec{q}_1, \vec{q}_2, \vec{q}_3) + \frac{\omega_c}{2} \sum_{i=1}^3 |\vec{q}_i|^2, \quad (1)$$

where the interaction has the symmetry

$$V(\vec{q}_1, \vec{q}_2, \vec{q}_3) = V(\mathcal{R}\vec{q}_1 + \vec{a}, \mathcal{R}\vec{q}_2 + \vec{a}, \mathcal{R}\vec{q}_3 + \vec{a}) \quad (2)$$

for arbitrary translations \vec{a} and rotations \mathcal{R} in the plane. A well-known transformation leads to new sets of canon-

ical variables: the kinematical momenta $\vec{\pi}_i = m\vec{v}_i$, and the so-called guiding centers $\vec{R}_i = \vec{q}_i - \hat{z} \times \vec{\pi}_i / (eB)$, which

have the following Poisson brackets

$$\{\pi_{i,\alpha}, \pi_{i,\beta}\} = \epsilon_{\alpha\beta} \delta_{ij} eB \quad (3a)$$

$$\{R_{i,\alpha}, R_{j,\beta}\} = -\epsilon_{\alpha\beta} \delta_{ij} (eB)^{-1} \quad (3b)$$

$$\{\pi_{i,\alpha}, R_{j,\beta}\} = 0 \quad (3c)$$

where $\epsilon_{\alpha\beta}$ is the antisymmetric tensor in two dimensions with $\epsilon_{12} = 1$. As $|B|$ becomes large, the cyclotron radii $r_i = |\vec{\pi}_i|/|eB|$ become far smaller than the scale at which the potential varies, and the $\vec{\pi}_i$ and \vec{R}_i decouple. The guiding center motion is then well described by the Hamiltonian

$$H_{\text{gc}}(\underline{x}, \underline{y}) = V[(x_1, y_1), (x_2, y_2), (x_3, y_3)] + \frac{\omega_c(|\underline{x}|^2 + |\underline{y}|^2)}{2}, \quad (4)$$

where the vectors $\underline{x} = (x_1, x_2, x_3)$ and $\underline{y} = (y_1, y_2, y_3)$ are the x and y components of the guiding centers in units chosen so as to render them canonically conjugate:

$$\{x_i, y_j\} = \delta_{ij}, \quad \{x_i, x_j\} = \{y_i, y_j\} = 0. \quad (5)$$

From the Poisson brackets, it is readily seen that

$$T_y = \sum_{i=1}^3 x_i, \quad T_x = \sum_{i=1}^3 y_i, \quad J = \frac{1}{2} \sum_{i=1}^3 (x_i^2 + y_i^2), \quad (6)$$

generate translations in y , translations in x , and rotations about the origin, all of which are symmetries of the interaction potential. Moreover, the harmonic external potential is proportional to J , which has vanishing Poisson bracket with the scalar $T_x^2 + T_y^2$. We thus find two independent integrals of the motion in involution, which for later convenience, can be traded for functions representing the orbital and spin angular momenta

$$L = \frac{1}{6} (T_x^2 + T_y^2), \quad S = J - L. \quad (7)$$

Since L , S and the Hamiltonian (4) are three integrals in involution, we conclude that the system is integrable.

To better understand the motion in the six-dimensional phase space, we represent the configuration of the system by the triangle defined by the three particles. Due to the harmonic potential in the Hamiltonian 4, the centroid coordinates $(T_y/3, T_x/3)$ of the triangle execute uniform circular motion about the origin with angular frequency ω_c (we take the positive sense of rotation as clockwise). This motion decouples from that of the relative coordinates, which describe the shape and orientation of the triangle. These can be conveniently represented in terms of the *spinor*

$$\Psi = \frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{3}(z_2 - z_1) \\ z_2 + z_1 - 2z_3 \end{pmatrix}, \quad (8)$$

where $z_i = x_i + iy_i$. As is easily verified, the spinor components satisfy the Poisson bracket relations

$$\{\Psi_\alpha, \Psi_\beta\} = \{\Psi_\alpha^*, \Psi_\beta^*\} = 0, \quad \{\Psi_\alpha^*, \Psi_\beta\} = i\delta_{\alpha\beta}, \quad (9)$$

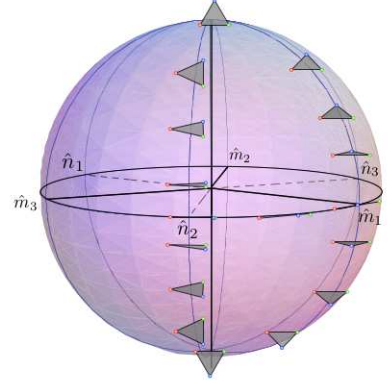


FIG. 1. Bloch sphere representation of the triangle's shape, with the corresponding shapes for two meridians passing through \hat{m}_1 and \hat{n}_2 .

and have vanishing Poisson brackets with T_x and T_y (and hence L). The normalization of the spinor is $\Psi^\dagger \Psi = S$, the conserved spin angular momentum, which is proportional to the square of the radius of gyration of the triangle. A phase change $\Psi \rightarrow e^{-i\chi} \Psi$ is equivalent to $(z_i - z_j) \rightarrow (z_i - z_j)e^{-i\chi}$ and hence to rotating the triangle by χ . Hence, the equivalence class $[\Psi] = \{z\Psi | z \in \mathbb{C} \setminus \{0\}\}$ corresponds to the triangle's *shape*, and the Bloch sphere for the unit spinors Ψ/\sqrt{S} is the space of possible shapes. To map shapes onto the sphere, define the unit vector

$$\vec{\xi} = \frac{1}{S} \Psi^\dagger \cdot \vec{\sigma} \Psi, \quad (10)$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of Pauli matrices [10]. Now let ρ_k be the squared length of the side of the triangle opposite to particle k and A be the triangle's *signed area* $A = [(y_3 - y_1)(x_2 - x_1) - (y_2 - y_1)(x_3 - x_1)]/2$. Then these quantities are given by:

$$\rho_k = 2S \left(1 + \vec{m}_k \cdot \vec{\xi} \right) \quad (1 \leq k \leq 3) \quad (11a)$$

$$\vec{m}_k = \left(\sin \frac{2\pi k}{3}, 0, \cos \frac{2\pi k}{3} \right) \quad (11b)$$

$$A = \frac{\sqrt{3}}{2} S \xi_2 \quad (11c)$$

We can now identify special classes of triangles on the Bloch sphere (see Fig. 1). Calling the intersection of the y -axis with the sphere the North pole, and the great circle perpendicular to it the equator, we see that the poles are triangles of maximal area, and hence equilateral, while points on the equator, being perpendicular to σ_2 , have zero area. An isosceles triangle, say with $\rho_1 = \rho_2$, is perpendicular to $\vec{m}_1 - \vec{m}_2$, and hence lies on the great circle connecting \vec{m}_3 and the pole. Therefore, the three great circles through the poles and the \vec{m}_k describe isosceles triangles. The \vec{m}_k and their antipodes—denoted by \bar{n}_k —are thus simultaneously isosceles and *collinear* triangles,

the \vec{m}_k with two identical sides adding up to the longest side, the \vec{n}_k with two coincident vertices.

In these new coordinates, the Hamiltonian is given by

$$H_{\text{rel}} = V(\Psi^\dagger \Psi, \Psi^\dagger \cdot \vec{\sigma} \Psi) + \omega_c \Psi^\dagger \Psi = V(S, \vec{\xi}) + \omega_c \xi, \quad (12)$$

where S and $\vec{\xi}$ are as defined earlier. From the Poisson brackets (9) and Hamilton's equations $\dot{\Psi}_\alpha = \{\Psi_\alpha, H\}$, we find that the spinor satisfies the equation of motion

$$i\dot{\Psi} = \left[\omega_c + \left(\frac{\partial V}{\partial S} \right)_{\vec{\xi}} + \frac{1}{S} \sum_{k=1}^3 \left(\frac{\partial V}{\partial \xi_k} \right)_S (\sigma_k - \xi_k \mathbb{1}) \right] \Psi, \quad (13)$$

where the subscripts in the derivatives indicate fixed variables. The equation of motion in shape space is then

$$\dot{\vec{\xi}} = \frac{2}{S} (\nabla_{\vec{\xi}} V)_S \times \vec{\xi}, \quad (14)$$

tracing orbits along the level curves of V on the Bloch sphere. Note that for homogeneous V of degree λ , $V(aS, \vec{\xi}) = a^\lambda V(S, \vec{\xi})$, the shape dynamics becomes independent of S when expressed in scaled time $\tilde{t} = S^{\lambda-1} t$.

So far we have looked at the dynamics of the rotation invariant characteristics of the triangle, namely S and $\vec{\xi}$. Analysis of the rotational motion is more subtle. While triangle rotations correspond to phase changes $\Psi \rightarrow e^{-i\chi} \Psi$, the notion of overall phase for Ψ is ill-defined. Still, a relative phase $d\chi = i\Psi^\dagger d\Psi / (\Psi^\dagger \Psi)$ can be defined between infinitesimally separated spinors Ψ and $\Psi + d\Psi$. Hence, a *dynamical* angular velocity can be defined to account for infinitesimal changes in orientation:

$$\omega_r^{(\text{dyn})} = \frac{1}{S} \Psi^\dagger \dot{\Psi} = \omega_c + \left(\frac{\partial V}{\partial S} \right)_{\vec{\xi}}. \quad (15)$$

Remarkably, when V is homogeneous of degree λ , $\omega_r^{(\text{dyn})} = \omega_c + \lambda V_0 / S$, where V_0 is the conserved potential energy. For finite times, the rotation angle is only well defined if the initial and final shapes are equal, that is, over a period T_s of the shape motion. One would then expect that $\Delta\chi = \int_0^{T_s} \omega_r^{(\text{dyn})} dt$ is the net rotation of the triangle, but this is incorrect. The reason is that an additional *geometric* phase, or Berry–Hannay phase [11–13], is acquired by parallel transport. Explicitly, if Ψ is parameterized as

$$\Psi = \sqrt{S} e^{-i\gamma} \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{-i\phi} \end{pmatrix}, \quad (16)$$

we find that $\omega_r^{(\text{dyn})} = \dot{\gamma} + \sin^2 \frac{\theta}{2} \dot{\phi}$. So the spinor acquires a phase $\Delta\gamma$, translating to the angular velocity $\omega_r = \frac{\Delta\gamma}{T_s}$:

$$\omega_r = \langle \omega_r^{(\text{dyn})} \rangle + \omega_r^{(\text{geo})}, \quad \omega_r^{(\text{geo})} \equiv -\frac{\Omega(E, S)}{2T_s} \quad (17)$$

where $\langle \omega_r^{(\text{dyn})} \rangle$ is the time average of $\omega_r^{(\text{dyn})}$ in the period T_s and $\Omega(E, S) = 2 \oint d\phi \sin^2 \frac{\theta}{2}$ is the oriented solid angle

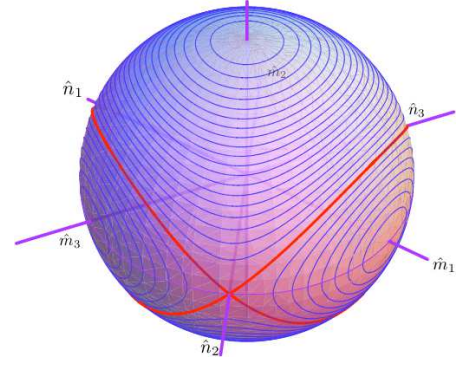


FIG. 2. Bloch sphere phase portrait for an interparticle central potential $u(r) \propto r^6$.

on the Bloch sphere enclosed by the level curve of E , as follows from Stoke's theorem. As expected, $\omega_r = \omega_r^{(\text{dyn})}$ at a fixed point of the shape motion.

Further analysis of the shape and rotational dynamics is possible in terms of action-angle variables, the details of which lie beyond the scope of this paper. Still, some salient features are worth mentioning. For the rotational motion, S is the natural action variable. The action variable I_s for the shape motion is obtained from the integral $\frac{1}{2\pi} \oint \underline{x} \cdot d\underline{y}$ [14], around the closed circuit in phase space given by a level curve of the energy on the Bloch sphere with fixed centroid. The final result has a nice geometric interpretation, namely

$$I_s = S \frac{\Omega(E, S)}{4\pi}. \quad (18)$$

This relation can be used to show that T_s , the characteristic period for the shape motion, is given by $T_s = \frac{S}{2} \frac{\partial \Omega(E, S)}{\partial E}$, and that ω_r in (17) is indeed the characteristic frequency for the rotational motion.

To fix ideas further, it may be useful to describe a special case in greater detail. Thus, if all three particles are identical and interact pairwise through a central potential of the form $u(r) \propto r^{2\lambda} = \rho^\lambda$, then one finds generically a phase portrait on the Bloch sphere with 8 critical points (Fig. 2), two of which are the north and south poles, which are always stable equilibria. The remaining six points are the \vec{m}_k and \vec{n}_k on the equator. For $\lambda > 2$, the \vec{m}_k are stable (elliptical) fixed points while the \vec{n}_k are unstable (hyperbolic) fixed points; for $\lambda < 2$ with $\lambda \neq 1$, on the other hand, the \vec{m}_k are unstable (hyperbolic) fixed points, whereas the \vec{n}_k are either stable fixed points ($0 < \lambda < 2$) or points at which the energy diverges ($\lambda < 0$); in the latter case nearby orbits encircle these points as if they were elliptical fixed points, and describe systems in which two particles revolve rapidly around each other in a tightly bound orbit, with a distant third particle slightly perturbing the motion, in a pattern somewhat analogous to that of the Sun–Earth–Moon system. The two exceptions to this pattern are

worthy of mention: First is the case $\lambda = 1$ ($u(r) \propto r^2$) for which $V \propto S$, so the triangle shape is frozen and ω_r is constant. The other exception is $\lambda = 2$ ($u(r) \propto r^4$), in which case $V \propto 6S^2(3 - \cos^2 \theta)$ where θ is the polar angle. In this case both the area and the radius of gyration of the triangle are constants of the motion, and for $\omega_c = 0$ both ω_r and ω_s scale linearly with this area, with ω_s/ω_r depends on θ and is independent of the size S .

We conclude with a brief discussion of the quantum mechanics of the problem. Re-interpreting the guiding center variables $\{x_i\}$ and $\{y_i\}$ as canonical operators with the Poisson brackets in Eq. (5) replaced by commutators (times i), we define the annihilation operators $a_i = \frac{1}{\sqrt{2}}(x_i + iy_i)$, so that $[a_i, a_j^\dagger] = \delta_{ij}$. To separate centroid and relative motion, define the operators

$$\begin{pmatrix} b \\ \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad (19)$$

and their adjoints, with $[b, b^\dagger] = [\Psi_\alpha, \Psi_\alpha^\dagger] = 1$ and all other commutators vanishing; in analogy with the classical case, the Ψ_α are interpreted as components of an operator-valued spinor. In terms of these, the orbital and spin angular momentum operators are

$$L = b^\dagger b + \frac{1}{2}, \quad S = \Psi^\dagger \Psi + 1, \quad (20)$$

with eigenvalues that are positive half-integers or integers respectively. The Ψ_α implement a Schwinger oscillator construction [15] of an $SU(2)$ algebra for the shape description. To see this, define

$$F_i = \frac{1}{2} \Psi^\dagger \cdot \sigma_i \Psi, \quad (21)$$

in terms of which the squared inter-particle distances are $\rho_k = 2S + 4\vec{F} \cdot \vec{m}_k$ and the triangle area is $A = \sqrt{3}F_2$. The F_i satisfy the commutation relations $[F_j, F_k] = i\epsilon_{jkl}F_l$, and thus commute with $F^2 = \vec{F} \cdot \vec{F}$, which is found to be

$$F^2 = \frac{S^2 - 1}{4}. \quad (22)$$

Therefore each eigenspace of S with eigenvalue s defines an s -dimensional spin- $(s-1)/2$ irreducible representation of $SU(2)$ for the F_i . With the previous symmetries of the interaction potential, the guiding center Hamiltonian can be written as $H_{gc} = V(S, \vec{F}) + \omega_c(L + S)$. The eigenvalues can then be labelled by the three quantum numbers of the problem: l, s and n , with $E_{l,s,n} = \tilde{E}_n^{(s)} + \omega_c(l + s)$, where the $\tilde{E}_n^{(s)}$ are the eigenvalues of the shape Hamiltonian for each sector s , namely, the reduced $s \times s$ matrix obtained from the potential with the F_i replaced by their respective representation matrices $F_i^{(s)}$:

$$H^{(s)} = V(s, \vec{F}^{(s)}). \quad (23)$$

Finally, to account for particle statistics, we note that $H^{(s)}$ can further be block-diagonalized into sectors transforming irreducibly under the action of the permutation group S_3 , from which the relevant symmetric or antisymmetric subspaces can be identified.

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